In [6] and [7] a computer oriented algorithm has been developed to find the solution to (34)-(36). At the present time the extension of this algorithm to the multiple structure constraint problem is being investigated.

REFERENCES

- M. Athans and P. Falb, Optimal Control. New York: McGraw-Hill, 1966.
- [2] D. L. Kleinman and M. Athans, "The design of suboptimal linear time-varying systems," *IEEE Trans. Automatic Contr.*, vol. AC-13, pp. 150–158, April 1968.
- [3] T. Chen, "Sensitivity analysis and optimum design of multivariable regulators with application to VTOL aircraft stabilization," Ph.D. dissertation, University of Pennsylvania, Philadelphia, 1968.
- [4] L. Zadeh and C. Desoer, Linear System Theory. New York: McGraw-Hill, 1963.
- [5] R. L. Kosut, "Suboptimal control of linear time-invariant multivariable systems subject to control structure constraints," Ph.D. dissertation, University of Pennsylvania, Philadelphia, 1969.
- [6] W. S. Levine and M. Athans, "On the determination of the optimal constant output-feedback gains for linear multivariable systems," *Proc. 3rd Ann. Princeton Conf. Inform. Sci. and Syst.*, 1969.

[7] A. Jameson and M. Rossi, "Optimization of the average response of linear systems of constrained configuration," Grumman Aerodynamics, Bethpage, N.Y., Tech. Rep. 393-68-7, November 1968.



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On the Input and Output Reducibility of Multivariable Linear Systems

MICHAEL HEYMANN

Abstract—By introducing into a constant linear system (F, G, H) with input vector u and output vector y an open-loop control u = Pv and observer z = Qy, a new constant linear system (F, GP, QH) results which has input vector v and output vector z. The problem investigated is one of constructing (F, GP, QH) so that v and z have minimal dimension, subject to the condition that the controllability and observability properties of (F, G, H) are preserved. It is shown that when the scalar field \mathscr{F} (over which the system is defined) is infinite, the minimal dimensions of v and z are essentially independent of the specific values of the input and output matrices G and H. It is also shown that this is not the case when \mathscr{F} is finite. Furthermore, an algorithm is presented for the construction of the minimal input (minimal output) (F, GP, QH), which is directly represented in a useful canonical form.

I. INTRODUCTION

CONSIDER a constant linear system given by a triple (F, G, H), where F, G, and H are constant $n \times n$, $n \times m$, and $p \times n$ matrices $(m, p \le n)$ with scalars in an arbitrary field \mathcal{F} . Thus (F, G, H) will be considered as the

basic data required to describe systems of either of two types:

1) a (continuous-time) constant real $(\mathcal{F} = \mathcal{R})$ linear dynamical system of the form

$$\frac{dx}{dt} = Fx + Gu$$
$$y = Hx$$

relating a real input vector u = u(t) to a real output vector v = v(t) through a real state vector x = x(t);

2) a (discrete-time) constant linear system of the form

$$x(k) = Fx(k - 1) + Gu(k - 1)$$
$$y(k) = Hx(k)$$

where $k = 1, 2, \dots$, is the (discrete) time variable and the scalar field is arbitrary. Thus all entries in the vectors and matrices are in \mathscr{F} and all operations are carried out in this field. (When \mathscr{F} is finite, e.g., the integers modulo a prime, this system is frequently referred to as a finite state machine.)

Let $Q_c \equiv [G, FG, \dots, F^{n-1}G]$ be the $n \times mn$ controllability matrix of (F, G, H), and let $Q_o \equiv [H^T, F^TH^T, \dots, (F^T)^{n-1}H^T]$ be its $n \times pn$ observability matrix. The

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controllable subspace of (F, G, H), denoted by S_c , is the subspace of the state space V^n (*n*-dimensional vector space over \mathscr{F}) spanned by the columns of Q_c . Thus S_c is the largest subspace of V^n that the control u can influence. Similarly, the observable subspace S_o of (F, G, H) is the linear span of the columns of Q_o , and it is the largest subspace of V^n in which the state influences the output.

Consider the open-loop control for (F, G, H) given as u = Pv, where v is a new input vector of dimension $k \le m$ and P is an $m \times k$ constant matrix with entries in \mathcal{F} . Similarly, let z = Qy be a new output device, where z is a new output vector of dimension $\hat{k} \le p$ and Q is a $\hat{k} \times p$ constant matrix with elements in \mathcal{F} .

It is required that the controllable and observable subspaces of the resultant system (F, GP, QH) be the same as the controllable and observable subspaces, respectively, of the original system (F, G, H). In applications it may be desirable to introduce such new input and output devices with the additional requirement that k and \hat{k} be as small as possible. Therefore, the following questions are of interest with regard to the preceding setup and the minimality of k and \hat{k} .

1) Given a state transition matrix F, what are the minimal dimensions k and \hat{k} for which there exist $n \times k$ and $\hat{k} \times n$ matrices G and H so that the resulting system (F, G, H) is both controllable and observable?

2) Consider a system (F, G, H) with controllable subspace S_c and observable subspace S_o . Under what conditions are the minimal values for k and \hat{k} dependent only on F and on the F invariants of S_c and S_o ? (By the F invariants of the F-invariant subspaces S_c and S_o we are referring to the invariant polynomials φ_i of the restrictions of the decomposition in Theorem 1 (Section II) to S_c and S_o , respectively.)

3) Given the minimal values for k and \hat{k} , how can the matrices P and Q be constructed?

Question 1), which is elementary (see discussion in Section II and also Theorem 2, Section III), was investigated by Vogt and Cullen [1] for the case when $\mathscr{F} = \mathscr{R}$. This question was also fully answered by Kalman [2] in a more general and abstract context. Vogt and Gunderson [3] discussed, for the special case when $\mathscr{F} = \mathscr{R}$, some of the problems which are studied for the general case here. However, even in that case, their analysis is incomplete since it ignores question 2) (which is most crucial for the validity of the whole paper-see specifically the first paragraph of Section III) and takes for granted an affirmative answer to it. As is shown in the present paper, question 2) has an affirmative answer only when the underlying field \mathcal{F} is infinite, and the proof of this fact is by no means trivial (even when $\mathscr{F} = \mathscr{R}$). Albertson and Womack [4] recently investigated the problem of finding, among the columns of G (rows of H), a minimal subset which preserves controllability (observability) when $\mathscr{F} = \mathscr{R}$. This is a subproblem of question 2), in which the columns of P (rows of Q) are restricted to be unit vectors (natural basis elements). Clearly, the number of vectors in such a subset is usually larger than $k(\hat{k})$ and coincides with $k(\hat{k})$ only in very special cases, even when $\mathcal{F} = \mathcal{R}$.

The contribution of the present paper is in fully analyzing and answering questions 2) and 3). Specifically, it is shown that when the field F is infinite, 2) can always be answered affirmatively; i.e., the minimal values of k and \hat{k} depend only on F and on the F invariants of S_c and S_o . Moreover, when (F, G, H) is controllable and observable, then the minimal k and \hat{k} are dictated only by F. In the case when \mathcal{F} is a finite field, no general statement can be made, as is exhibited by an example. (Although the finite field case is not further investigated, it can be shown that the main results for infinite fields are also valid for finite fields, provided the characteristic of the field is larger than the dimension n of the state space V^n .) Also, an algorithm is developed for the construction of P (a similar algorithm is valid for Q) that simultaneously transforms the resultant system (F, GP, H) into a useful canonical form which resembles one of the canonical forms presented by Luenberger [5].

The paper is organized as follows. In Section II certain preliminary concepts are defined. (For background material in linear algebra the reader is referred to Gantmacher [6].) The main results on input and output reducibility are given in Section III and are further developed in Section IV, where an algorithm is presented for construction of P which immediately represents the resultant system in canonical form. Some generalizing remarks are presented in Section V and the paper is concluded in Section VI. A fundamental theorem on which the main results of the present paper hinge is stated and proved in the Appendix.

II. PRELIMINARIES

Let F be a linear operator in an n-dimensional vector space V^n over an arbitrary field \mathcal{F} . Let I be an F-invariant subspace of V^n , and let the set of vectors $g_1, \dots, g_m \in I$ be a generating set for I; i.e., every vector $g \in I$ can be expressed as a linear combination of vectors of the form F^jg_i , where $i = 1, \dots, m$ and $j = 0, 1, 2, \dots$. Let C be the collection of all sets of vectors generating I. (C is clearly not empty since it contains every basis of I.) Generating sets in the collection C which contain the smallest number of component vectors are called minimal generating sets for I.

The following well-known theorem of linear algebra is of interest here.

Theorem 1¹

Relative to a given linear operator F, the space V^n (over an arbitrary field \mathscr{F}) can be decomposed into a direct sum of cyclic subspaces I_1, I_2, \dots, I_m with minimal polynomials $\varphi_1, \varphi_2, \dots, \varphi_m$, respectively, such that

$$V^n = I_1 \oplus I_2 \oplus \cdots \oplus I_m.$$

Moreover, φ_1 coincides with ψ , the minimal polynomial of V^n , and for each *i*, φ_{i+1} divides φ_i .

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<sup>1</sup> See [6, p. 187, theorem 3].
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The standard proof of Theorem 1 is based on the exhibition of a cyclic generating set of vectors for V^n , i.e., a set of vectors g_1, \dots, g_m with the property that g_i generates I_i , $i = 1, \dots, m$. A cyclic generating set is clearly also a minimal generating set for V^n . (In fact, g_1 generates a cyclic subspace of V^n of highest possible dimension, and each succeeding g_i generates a cyclic subspace of highest dimension of V^n modulo the sum of the preceding cyclic subspaces.)

Let k_1, \dots, k_m be the dimensions of the cyclic subspaces I_1, \dots, I_m in Theorem 1, and let $\{g_1, \dots, g_m\}$ be a cyclic generating set for V^n (relative to the operator F). Let $\alpha_1, \dots, \alpha_m$ be a set of vectors such that

$$\alpha_1 = g_1$$

$$\alpha_2 \equiv g_2 \pmod{I_1}$$

$$\vdots$$

$$\alpha_m \equiv g_m \pmod{I_1 + \dots + I_{m-1}}.$$

It is evident that α_1 generates I_1 and, for each i > 1, α_i generates $I_i \pmod{I_1 + \cdots + I_{i-1}}$. Thus the vectors

$$\alpha_1, F\alpha_1, \cdots, F^{k_1-1}\alpha_1, \alpha_2, \cdots, F^{k_2-1}\alpha_2, \cdots, \alpha_m, \cdots, F^{k_m-1}\alpha_m$$

are linearly independent and form a basis for V^n . The set of vectors $\{\alpha_1, \dots, \alpha_m\}$ is called a *semicyclic generating set* for V^n and the basis a *semicyclic basis*. Clearly, a semicyclic generating set for V^n is a minimal generating set.

It can be readily verified by direct computation that in a semicyclic basis for V^n the operator F is given by the matrix F_I :

$$F_{I} = \begin{bmatrix} F_{I_{1}} & F^{12} & \cdots & F^{1m} \\ 0 & F_{I_{2}} & \cdots & F^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{Im} \end{bmatrix}$$

where, for $i = 1, \dots, m$, F_{I_i} is the companion matrix of $\varphi_i = \lambda^{k_i} + a_{i1}\lambda^{k_i-1} + \alpha_{ik_i}$ (of Theorem 1) given by

$$F_{I_i} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_{ik_i} \\ 1 & 0 & \cdots & 0 & -a_{i,k_i-1} \\ 0 & 1 & \cdots & 0 & -a_{i,k_i-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{i1} \end{bmatrix}$$

and F^{ij} (j > i) are matrices with zero elements except (possibly) in the last columns. In the case when a semicyclic basis is also a cyclic basis, then the submatrices F^{ij} are identically zero.

III. INPUT AND OUTPUT REDUCIBILITY

Consider a linear system (F, G, H) with controllable subspace S_c and observable subspace S_o . It is evident from the discussion in Section II that the columns g_1, \dots, g_m form (relative to F) a generating set for the controllable subspace,

and similarly, the rows of H form (relative to F^T) a generating set for the observable subspace. Let $\mathscr{L}(G)$ denote the linear span of the columns of G and $C_{\mathscr{L}(G)}[S_c]$ be the collection of all generating sets of S_c which are contained in $\mathscr{L}(G)$. Clearly, $C_{\mathscr{L}(G)}[S_c]$ is a subset of $C[S_c]$, the collection of all generating sets for S_c . Similarly, define $\mathscr{L}(H^T)$, $C_{\mathscr{L}(H^T)}[S_o]$, and $C[S_o]$. The central questions dealt with in the present paper can now be stated precisely as follows.

1) Under what conditions does $C_{\mathscr{L}(G)}[S_c]$ contain a minimal generating set for S_c ; i.e., under what conditions is the size of the smallest generating set for S_c which is contained in $C_{\mathscr{L}(G)}[S_c]$ equal to the size of the smallest element in $C[S_c]$? (An equivalent question is also posed regarding $C_{\mathscr{L}(H^T)}[S_o]$ and $C[S_o]$).

2) How can such a minimal generating set be constructed?

It is well known (see Kalman [7]) that (F, G, H) can be split into a controllable (observable) subsystem and an uncontrollable (unobservable) subsystem so that the controllable subspace S_c (the observable subspace S_o) of the original system essentially constitutes the whole state space in the controllable (observable) subsystem. Furthermore, the system can be split into four interconnected subsystems only one of which is both controllable and observable, whereas the others are either completely uncontrollable or unobservable or both. Henceforth it will be assumed, without loss of generality, that the appropriate decomposition has already been performed and that the system under consideration is controllable and/or observable as the case may require.

The following main result of this paper is a direct consequence of Theorem 4 (see the Appendix).

Theorem 2

Let (F, G, H) be a constant linear system over an infinite field \mathscr{F} , where F, G, and H are $n \times n$, $n \times m$, and $p \times n$ matrices, respectively. Let the minimal generating sets for the state space V^n contain k vectors (i.e., V^n splits according to Theorem 1 into k cyclic invariant subspaces I_1, \dots, I_k). Then:

a) (F, G, H) is controllable if and only if $m \ge k$ and there exist k vectors $\tilde{g}_1, \dots, \tilde{g}_k \in \mathscr{L}(G)$ such that the system (F, \tilde{G}, H) is controllable, where \tilde{G} is the $n \times k$ matrix $[\tilde{g}_1, \dots, \tilde{g}_k]$.

b) (F, G, H) is observable if and only if $p \ge k$ and there exist k vectors $\tilde{h}_1, \dots, \tilde{h}_k \in \mathscr{L}(H^T)$ such that (F, G, \tilde{H}) is observable, where \tilde{H} is the $k \times n$ matrix $[\tilde{h}_1, \dots, \tilde{h}_k]^T$.

It is clear in view of Theorem 4 in the Appendix and the example following it that Theorem 2 is generally invalid for systems defined over finite fields, and, in particular, it is not valid for finite state machines.

Several interesting observations can be made with regard to Theorem 2.

1) Let (F, G, H) be a controllable and observable system over an infinite field. Then the *m*-dimensional control vector *u* can be replaced by a new *k*-dimensional control vector *v*, which is related to *u* by u = Pv, with *P* an $m \times k$ matrix of rank *k*. Similarly, the output vector *y* can be replaced by a k-dimensional vector z such that z = Qy, with Q a $k \times p$ matrix of rank k. The important consequence of Theorem 2 is that there exist *proper* choices of P and Q so that the resultant system (F, GP, QH) is controllable and observable.

2) The new control vector v can be chosen so that the system is split into k unidirectionally connected subsystems, each of which is controllable by a *different* single component of v. A similar choice can be made for the output vector z with regard to observability. This observation is based on the fact that (see proof of Theorem 4) the vectors $\tilde{g}_1, \dots, \tilde{g}_k$ and the vectors $\tilde{h}_1, \dots, \tilde{h}_k$ can be chosen to form semicyclic generating sets for V^n .

This last fact will be further studied in the next section, where a corollary to Theorem 2 is presented to the effect that the system (F, GP, QH) can be reduced directly into certain useful canonical forms.

IV. CANONICAL FORMS OF REDUCED SYSTEMS

From this section on it will be assumed throughout that the field \mathscr{F} is infinite. The following corollary to Theorem 2 is easily verified by the use of the constructive proof of Theorem 4 and direct computation.

Corollary 1

a) If the system of Theorem 2 is controllable, then there exists a transformation $P(m \times k \text{ matrix})$ on the input space and a nonsingular coordinate transformation $T(n \times n \text{ matrix})$ on the state space such that the system (TFT^{-1}, TGP, HT^{-1}) is controllable and the matrices TFT^{-1} and TGP have the canonical representations

$$TFT^{-1} = F_{I} = \begin{bmatrix} F_{I_{1}} & 0 & \cdots & 0 \\ 0 & F_{I_{2}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & F_{I_{k}} \end{bmatrix}$$
$$TGP = T\tilde{G} = \tilde{G}_{I} = \begin{bmatrix} \tilde{G}_{I_{1}} \\ \vdots \\ \tilde{G}_{I_{k}} \end{bmatrix}$$

where, for $j = 1, \dots, k$, F_{I_j} is the companion matrix of φ_j (the minimal polynomial of I_j) as given in Section II, and where, for $j = 1, \dots, k$, \tilde{G}_{I_j} is a $p_j \times k$ matrix (p_j is the dimension of the *j*th invariant subspace) given as

$$\tilde{G}_{I_j} = \begin{bmatrix} \tilde{G}_{I_j}^{11} & \tilde{G}_{I_j}^{12} \\ \tilde{G}_{I_j}^{21} & \tilde{G}_{I_j}^{22} \end{bmatrix}$$

where $\tilde{G}_{I_j}^{11}$ is a $1 \times j$ matrix of the form $[0, \dots, 0, 1]$, $\tilde{G}_{I_j}^{12}$ is a $1 \times (k - j)$ zero matrix, $\tilde{G}_{I_j}^{21}$ is a $(p_j - 1) \times j$ zero matrix, and $\tilde{G}_{I_j}^{22}$ is a $(p_j - 1) \times (k - j)$ matrix with unspecifiable² elements.

b) If the system of Theorem 2 is observable, then there exists a transformation Q ($k \times p$ matrix) on the output space and a (nonsingular) coordinate transformation \hat{T} ($n \times n$ matrix) on the state space such that the system ($\hat{T}F\hat{T}^{-1}, \hat{T}G, QH\hat{T}^{-1}$) is observable and the matrices $\hat{T}F\hat{T}^{-1}$ and $QH\hat{T}^{-1}$ have the canonical representations

$$\hat{T}F\hat{T}^{-1} = F_I^T = \begin{bmatrix} F_{I_1}^T & 0 & \cdots & 0\\ 0 & F_{I_2}^T & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & F_{I_k}^T \end{bmatrix}$$
$$QH\hat{T}^{-1} = \tilde{H}\hat{T}^{-1} = \tilde{H}_I = [\tilde{H}_{I_1}, \cdots, \tilde{H}_{I_k}]$$

where, for $j = 1, \dots, k$, $F_{I_j}^T$ is the transpose of F_{I_j} , the companion matrix of φ_j as given in Section II, and \tilde{H}_{I_j} is a $k \times p_j$ matrix (p_j is the dimension of the *j*th invariant subspace) given as

$$\tilde{H}_{I_j} = \begin{bmatrix} \tilde{H}_{I_j}^{11} & \tilde{H}_{I_j}^{12} \\ \tilde{H}_{I_j}^{21} & \tilde{H}_{I_j}^{22} \end{bmatrix}$$

where $\tilde{H}_{I_j}^{11}$ is a $j \times 1$ matrix of the form $[0, \dots, 0, 1]^T$, $\tilde{H}_{I_j}^{12}$ is a $j \times (p_j - 1)$ zero matrix, $\tilde{H}_{I_j}^{21}$ is a $(k - j) \times 1$ zero matrix, and $\tilde{H}_{I_j}^{22}$ is a $(k - j) \times (p_j - 1)$ matrix with unspecifiable elements.

The following are some pertinent remarks regarding this corollary.

1) In a controllable and observable system (F, G, H), one cannot, in general, find T, P, Q so that both $T\tilde{G}$ and $\tilde{H}T^{-1}$ are simultaneously in canonical form; that is, if $\tilde{T}G$ is in canonical form, then $\tilde{H}T^{-1}$ is generally not, and vice versa.

2) The transformations T and P or \hat{T} and Q are not unique and neither are the canonical representations; i.e., there exist many choices of transformations T and P or \hat{T} and Q, each of which leads to different elements in $\tilde{G}_{I_j}^{22}$ or in $\tilde{H}_{I_j}^{22}$. However, the number of unspecifiable elements is independent of the choice of T and P or \hat{T} and Q.

3) The construction of the canonical representation is nontrivial: there exist transformations T that take F into F_I for which there do not exist P that transform TG into \tilde{G}_I (and similarly for \hat{T} and Q).

An algorithm will now be developed for construction of T and P that transform a controllable system (F, G, H) into a canonical representation of Corollary 1a). A similar algorithm can be developed for construction of \hat{T} and Q that transform an observable system into canonical form. The details of this algorithm are left to the reader.

It is convenient to take F in its Jordan canonical form, that is,

$$F = \text{diag}\left[J(\lambda_1, r_{11}), J(\lambda_2, r_{12}), \cdots, J(\lambda_q, r_{1q}), \cdots, J(\lambda_q, r_{kq})\right]$$

where

$$\sum_{i=1}^{k} \sum_{j=1}^{q} r_{ij} = n = \dim(F)$$

 $^{^{2}}$ These elements are unspecifiable in the sense that they depend both on the system parameters and the particular transformations T and P chosen.

the λ_j , $j = 1, \dots, q$, are the distinct (complex) eigenvalues of F, $J(\lambda, r)$ is the $r \times r$ Jordan block with eigenvalue λ

$$J(\lambda, r) = \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & \ddots & \lambda \end{bmatrix}$$

and

 $\dim (J(\lambda_j, r_{ij})) \ge \dim (J(\lambda_j, r_{i+1,j})), \quad i = 1, \cdots, k-1.$

Necessary and sufficient conditions will now be stated for a vector g to have ψ (the minimal polynomial of V^n) as its minimal polynomial. First the special case where k = 1(cyclic space) is considered.

Lemma 1

Let F be a linear operator on V" in its Jordan form, with eigenvalues $\lambda_1, \dots, \lambda_n$, i.e.,

$$F = \operatorname{diag} \left[J(\lambda_1, r_1), J(\lambda_2, r_2), \cdots, J(\lambda_q, r_q) \right]$$

such that V^n is cyclic and has a minimal polynomial ψ . Then a vector $g = (g_1, \dots, g_n)$ (in the same basis as F) has ψ as its minimal polynomial if and only if the components $g_{r_1}, g_{r_1+r_2}, \dots, g_n$ of g are nonzero.

Remark: A result related to Lemma 1 stating necessary and sufficient conditions for controllability of a system (F, G, H) when F is in Jordan canonical form was stated by Kalman [8] and later restated and proved by Chen and Desoer [9]. With slight modification their proof applies also to this lemma.

Now the general case when $k \ge 1$ (i.e., when the space is not necessarily cyclic) is considered.

Theorem 3

Let F be a linear operator on V^n in its Jordan form, with eigenvalues $\lambda_1, \dots, \lambda_q$ and minimal polynomial ψ . Then a vector $g = (g_1, \dots, g_n)$ has ψ as its minimal polynomial if and only if, for each λ_j , $j = 1, \dots, q$, the component of g corresponding to the last row of at least one³ Jordan block of λ_i of maximal dimension is nonzero.

Proof: The proof follows readily from the fact that V^n splits into the direct sum of cyclic invariant subspaces (Theorem 1) and from Lemma 1.

The following is an algorithm for construction of the transformations T and P that take a controllable linear system (F, G, H) into the canonical representation of Corollary 1a). The algorithm is valid in view of the preceding theory and specifically Lemma 2 and Theorems 3 and 4. Certain details which do not follow directly from the pre-

³ Note that there may be more than one Jordan block of maximal dimension for a given eigenvalue.

ceding theorems can be readily verified by direct computation.

Algorithm

Let (F, G, H) be a controllable linear system. Let F have eigenvalues $\lambda_1, \dots, \lambda_q$, and let V^n split into k cyclic invariant subspaces (i.e., the highest multiplicity of Jordan blocks for any single eigenvalue is k). Assume that F is in Jordan form

$$F = \operatorname{diag}\left[J(\lambda_1, r_{11}), J(\lambda_2, r_{12}), \cdots, J(\lambda_a, r_{1a})\right], \cdots, J(\lambda_a, r_{ka})\right]$$

where

$$\dim \left(J(\lambda_j, r_{ij})\right) \ge \dim \left(J(\lambda_j, r_{i+1,j})\right), \quad i = 1, \cdots, k-1$$

and

$$\sum_{i=1}^{k} \sum_{j=1}^{q} r_{ij} = \sum_{i=1}^{k} p_i = n$$

where p_i is the dimension of the *i*th cyclic subspace.

Step 1:

a) Construct a vector $\hat{g}_1 = \sum_{i=1}^m \alpha_{i1}g_i$, where the g_i are the columns of G and $\alpha_{11} \neq 0$, such that in \hat{g}_1 the r_{11} th, $(r_{11} + r_{12})$ th, \cdots , $(r_{11} + \cdots + r_{1q})$ th components are all nonzero.

b) Construct the $n \times n$ (nonsingular) linear transformation T_1 such that

$$T_1^{-1} = [\hat{g}_1, F\hat{g}_1, \cdots, F^{p_1 - 1}\hat{g}_1, e_{p_1 + 1}, \cdots, e_n]$$

where e_j is the jth column of the n-dimensional unit matrix.
c) Obtain F₁ and G₁ as

$$F_{1} = T_{1}FT_{1}^{-1} = \text{diag}[F_{I_{1}}, J(\lambda_{1}, r_{21}), \cdots, J(\lambda_{q}, r_{2q}), \cdots, J(\lambda_{q}, r_{kq})]$$

where F_{I_1} is the companion matrix of the minimal polynomial $\varphi_1 = \psi$ of V^n and

$$G_1 = T_1 \hat{G}_1 = [T_1 \hat{g}_1, T_1 g_2, T_1 g_3, \cdots, T_1 g_m]$$

where $\hat{G}_1 = [\hat{g}_1, g_2, \cdots, g_m]$.

Step $l (l = 2, \cdots, k)$:

a) Construct a vector $\hat{g}_l = \sum_{i=1}^m \alpha_{il} (T_1 \cdots T_{l-1}) g_i$, where $a_{ll} \neq 0$, such that in \hat{g}_l the $(p_1 + \cdots + p_{l-1} + r_{l1})$ th, $(p_1 + \cdots + p_{l-1} + r_{l1} + r_{l2})$ th, \cdots , $(p_1 + \cdots + p_l)$ th components are nonzero.

b) Construct the $n \times n$ (nonsingular) linear transformation T_l such that

$$T_{l}^{-1} = [e_{1}, e_{2}, \cdots, e_{p_{l}-1}, \tilde{g}_{l}, F_{l-1}\tilde{g}_{l}, \cdots, F_{l-1}^{p_{l}-1}\tilde{g}_{l}, e_{p_{1}+\cdots+p_{l-1}}, \cdots, e_{n}]$$

where \tilde{g}_l is \hat{g}_l with the first $p_1 + \cdots + p_{l-1}$ components replaced by zero.

c) Obtain F_l and G_l as

$$F_{l} = T_{l}F_{l-1}T_{l}^{-1} = \text{diag}\left[F_{l_{1}}, \cdots, F_{l_{l}}, J(\lambda_{1}, r_{l+1,1}), \cdots, J(\lambda_{q}, r_{l+1,q}), \cdots, J(\lambda_{q}, r_{kq})\right]$$

where F_{I_l} is the companion matrix of φ_l , the minimal polynomial of $V^n \pmod{I_1 + \cdots + I_{l-1}}$, and

$$G_{l} = T_{l}\hat{G}_{l} = (T_{l}\cdots T_{1}\hat{g}_{1}, T_{l}\cdots T_{2}\hat{g}_{2}, \cdots,$$
$$T_{l}\hat{g}_{l}, T_{l}\cdots T_{1}g_{l+1}, \cdots, T_{l}\cdots T_{1}g_{m})$$

where

$$\hat{G}_{l} = (T_{l-1} \cdots T_{1} \hat{g}_{1}, \cdots, \hat{g}_{l}, T_{l-1} \cdots T_{1} g_{l+1}, \cdots, T_{l-1} \cdots T_{1} g_{m}).$$

Step k + 1:

Let $T = T_k \cdot T_{k-1} \cdots T_i$ and let P_1 be the matrix

$$P_{1} = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mk} \end{bmatrix}.$$

Then

$$F_k = F_I = TFT^{-1}$$

and

$$G'_I = TGP_1$$

where G'_I is the matrix composed of the first k columns of G_k with the *j*th column

$$g^{\prime j} = T_k \cdots T_j \hat{g}_j = (g_1^{\prime j}, \cdots, g_n^{\prime j}), \quad j = 1, \cdots, k,$$

being of the form

$$g'^1 = (1, 0, \cdots, 0)$$

and

$$g'^{j} = (g'^{j}_{1}, \cdots, g'^{j}_{p_{1}}, \cdots, p_{j}, 1, 0, \cdots, 0), \quad j = 2, \cdots, k.$$

Step k + 2:

Construct (in the obvious way) a $k \times k$ matrix P_2 so that $P = P_1 P_2$ and $\tilde{G}_I = G'_I P_2$.

V. FURTHER REMARKS

The procedure of finding k controls in the m-dimensional input space that preserve controllability of a system is called *input reduction*. The equivalent procedure on the output space is called *output reduction*. Clearly, a system (F, G, H) over an infinite field which is both controllable and observable can be both input reduced and output reduced. The resultant system (F, GP, QH) is called *externally reduced*: it possesses the smallest number of inputs and outputs for the given state space structure.

When the input and output reducing transformations are chosen according to Corollary 1, an externally reduced system (F, GP, QH) can be represented according to either canonical form by appropriate choices or coordinate transformations of the state space. When the canonical form of Corollary 1a) is chosen, QHT^{-1} will have arbitrary parameters; when the canonical form of Corollary 1b) is chosen, TGP will have arbitrary elements.

The transformation $T, P:(F, G) \to (F_I, G_I)$ is, in general, not an isomorphism and the systems (F, G) and (F_I, G_I) are not internally isomorphic since P takes an *m*-dimensional subspace of V^n into a $k(\leq m)$ -dimensional subspace of V^n . The same, of course, holds for $T, Q:(F, H) \to (F_I, H_I)$. It is possible, however, to extend the transformations Pand Q by adjoining m - k linearly independent columns to P and p - k linearly independent rows to Q. Thus, let $\overline{P} = [P, \widetilde{P}]$ and $\overline{Q} = \begin{bmatrix} Q \\ \widetilde{Q} \end{bmatrix}$ for \widetilde{P} and \widetilde{Q} so that \overline{P} is an $m \times m$ nonsingular matrix and \overline{Q} is a $p \times p$ nonsingular matrix. The matrices \overline{P} and \overline{Q} define coordinate changes in the input space and output space, respectively, and the transformations

where

and

where

$$\overline{G}_I = [G_I, TG\widetilde{P}]$$

 $T, \overline{P}: (F, G) \to (F_T, \overline{G}_T)$

$$T, \overline{O}: (F, H) \to (F_I, \overline{H}_I)$$

$$\overline{H}_{I} = \begin{bmatrix} H_{I} \\ \tilde{Q}HT^{-1} \end{bmatrix}$$

are isomorphisms. Such coordinate transformations may be very useful in a variety of computational applications of linear systems theory.

VI. CONCLUSION

It is shown in this paper that for a controllable and observable system (F, G, H) defined over an infinite field, a reduction of the input and output spaces without losing controllability or observability can be attained by taking appropriate linear combinations of the existing inputs and outputs: the minimal number of inputs and outputs in the reduced system (which can always be attained) is independent of the specific matrices G and H and is equal to the number of invariant factors of F. This reduction cannot be effected when the field is finite and is therefore invalid for certain types of finite state machines. An algorithm is also presented which systematically constructs the reducing transformations while simultaneously taking the resultant system into a canonical form.

Since many control problems depend exclusively on the properties of controllability and observability, the input and output reduction has both theoretical and practical applications. On the theoretical side, it may be assumed (frequently with considerable simplification of the analysis) that, given a controllable and observable system (\mathscr{F} infinite), it is reduced, i.e., it possesses the minimal number of inputs and outputs. It can further be assumed that the system is in canonical form, thus simplifying the computation. On the practical side, it can be expected that control hardware may be saved by lumping inputs (outputs) in the specified way. This may be particularly desirable when constructing feedback compensators.

Appendix

EXISTENCE OF MINIMAL GENERATING SETS

Let F be a linear operator on an n-dimensional vector space V^n and let g_1, \dots, g_k be a set of vectors in V^n with linear span $\mathscr{L}(G)$. Denote by $\psi_{\mathscr{L}(G)}$ the minimal polynomial of $\mathscr{L}(G)$, i.e., the monic polynomial φ of smallest degree such that $\varphi(F)v = 0$, for every vector $v \in \mathscr{L}(G)$. Thus $\psi_{\mathscr{L}(G)}$ is the minimal polynomial of the subspace of V^n generated by the set $\{g_1, \dots, g_k\}$. Also, for $v \in V^n$, denote by φ_v the minimal polynomial of v.

The following lemma, the proof of which has been given elsewhere [10], states that, except for the case when the underlying field is finite, the subspace $\mathscr{L}(G)$ always contains a vector v such that $\varphi_v = \psi_{\mathscr{L}(G)}$. (In fact, when \mathscr{F} is the field of the real or complex numbers, almost every vector $v \in \mathscr{L}(G)$ has this property.) Using this fact it will be shown (Theorem 4) that, except for the case of a finite field, one can always find in the linear span of a generating set of an invariant subspace I a minimal generating set.

Lemma 2

Let V^n be an *n*-dimensional vector space over an infinite field \mathscr{F} , let *F* be a linear operator on V^n , and let g_1, \dots, g_k be a set of vectors in V^n with linear span $\mathscr{L}(G)$. Then there exists $v \in \mathscr{L}(G)$ such that $\varphi_v = \psi_{\mathscr{L}(G)}$.

Theorem 4

Let V^n be an *n*-dimensional vector space over an infinite field \mathscr{F} , let F be a linear operator on V^n , and let $g_1, \dots, g_m \in V^n$ be a generating set of vectors for V^n .

Then there exists in $\mathscr{L}(G)$ a set of vectors $\tilde{g}_1, \dots, \tilde{g}_k$ which is a minimal generating set for V^n .

Proof: Let $\varphi_1, \dots, \varphi_m$ be the minimal polynomials of g_1, \dots, g_m and let ψ_1 be their least common multiple. Since the set g_1, \dots, g_m generates V^n , it is clear that $\psi_1 = \psi$, the minimal polynomial of V^n (relative to F). By Lemma 2 there exists a vector $\tilde{g}_1 \in \mathscr{L}(G)$ whose minimal polynomial $\varphi_{\tilde{g}_1}$ coincides with ψ_1 . If p_1 is the degree of ψ_1 , it follows that the vectors $\tilde{g}_1, F\tilde{g}_1, \cdots, F^{p_1-1}\tilde{g}_1$ are linearly independent and span a cyclic invariant subspace I_1 of V^n . Let ψ_2 be the minimal polynomial of $V^n \pmod{I_1}$ and let $\varphi_{1,I_1}, \cdots, \varphi_{n,I_n}$ φ_{m,I_1} be the minimal polynomials of $g_1, \dots, g_m \pmod{I_1}$. Then ψ_2 is the least common multiple of $\varphi_{1,I_1}, \cdots, \varphi_{m,I_1}$, and by Lemma 2 there exists a vector $\tilde{g}_2 \in \mathscr{L}(G)$ whose (relative) minimal polynomial (mod I_1) is ψ_2 . If p_2 is the degree of ψ_2 , we have $\tilde{g}_2, F\tilde{g}_2, \dots, F^{p_2-1}\tilde{g}_2$ span a cyclic invariant subspace I_2 of $V^n \pmod{I_1}$. Thus $\tilde{g}_2 \in I_1 + I_2$ and the $p_1 + p_2$ vectors $\tilde{g}_1, \dots, F^{p_1-1}\tilde{g}_1, \tilde{g}_2, \dots, F^{p_2-1}\tilde{g}_2$ are linearly independent.

This procedure can be continued until there are k vectors $\tilde{g}_1, \dots, \tilde{g}_k \in \mathscr{L}(G)$ (with $\tilde{g}_1 \in I_1, \tilde{g}_2 \in I_1 + I_2, \dots, \tilde{g}_k \in I_1 + I_2 + \dots + I_k$) such that $\tilde{g}_1, \dots, F^{p_1-1}\tilde{g}_1, \dots, g_k, \dots, F^{p_k-1}\tilde{g}_k$ are linearly independent vectors which span V^n (i.e., $\sum_{i=1}^k p_i = n$). Thus $\tilde{g}_1, \dots, \tilde{g}_k$ form a generating set for V^n . That this generating set is minimal follows from the fact that ψ_1, \dots, ψ_k are the invariant factors of F [6]. In fact, suppose there exists any smaller generating set for V^n :

it, contradicting the invariance of the polynomials ψ_i . This completes the proof.

That Theorem 4 is not valid for finite fields (a direct consequence of the fact that Lemma 2 is not valid in this case) is illustrated by the following example. Let V^4 be a (four-dimensional) vector space over Z_2 (the integers modulo 2). Let e_1, \dots, e_4 be a basis for V^4 and let F be a linear operator in V^4 such that $Fe_1 = e_2$, $Fe_2 = e_1 + e_2$, $Fe_3 = e_3$, $Fe_4 = 0$. Let g_1 , g_2 be vectors in V^4 , where $g_1 = e_1 + e_3$ and $g_2 = e_3 + e_4$. Then $\varphi_{g_1} = (x + 1)(x^2 + x + 1)$, $\varphi_{g_2} = x(x + 1)$, and, consequently, $\psi_{\mathscr{L}(G)}$ is the least common multiple $(\varphi_{g_1}, \varphi_{g_2}) = x(x + 1)(x^2 + x + 1)$. Thus the set $\{g_1, g_2\}$ generates V^4 , and so does the vector $v = e_2 + e_3 + e_4$. However, no single vector in $\mathscr{L}(G)$ generates V^4 since the only nonzero vectors in $\mathscr{L}(G)$ are g_1, g_2 and $g_3 = g_1 + g_2 = e_1 + e_4$ with $\varphi_{g_3} = x(x^2 + x + 1)$.

References

- W. G. Vogt and C. G. Cullen, "The minimum number of inputs required for the complete controllability of a linear stationary dynamical system," *IEEE Trans. Automatic Contr.* (Correspondence), vol. AC-12, p. 314, June 1967.
- [2] R. E. Kalman, "Algebraic theory of linear systems," Arch. Automat. Telemech. (Warsaw), vol. 11, pp. 119-129, 1966.
- [3] W. G. Vogt and R. W. Gunderson, "Canonical realizations," Proc. Ist Asilomar Conf. Circuits and Syst., November 1967, pp. 782–789.
- [4] H. D. Albertson and B. F. Womack, "Input and output degrees of freedom of linear systems," *IEEE Trans. Automatic Contr.* (Correspondence), vol. AC-13, p. 745, December 1968.
- [5] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automatic Contr.* (Short Papers), vol. AC-12, pp. 290–293, June 1967.
- [6] F. R. Gantmacher, The Theory of Matrices, vol. 1. New York: Chelsea, 1959.
- [7] R. E. Kalman, "Mathematical description of linear dynamical systems," SIAM J. Contr., vol. 1, pp. 152–192, 1963.
- [8] R. E. Kalman, "Irreducible realizations and the degree of a rational matrix," SIAM J. Appl. Math., vol. 13, pp. 520–544, 1965.
- [9] C. T. Chen and C. A. Desoer, "Controllability and observability of composite systems," *IEEE Trans. Automatic Contr.*, vol. AC-12, pp. 402-409, August 1967.
- [10] M. Heymann and J. A. Thorpe, "A note on minimal polynomials," Linear Algebra and Its Applications (to be published).



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